2 Euclid's Elements, Axioms, and Rigor

2.1 Proofs with logical gaps

In the previous section I gave several examples of geometric proofs. Important to understand that in *each* of these examples I had to use something, which I assumed to be true (be it the fact that the sum of the angles of a triangle is equal to π, or the comparison conditions for the triangle equality, or basic properties of natural numbers, etc). It should be clear that all these "true facts" also require to be proved, and so on. If one attempts to prove *everything* sooner or later some kind of *circular argument* will be involved, which will deem the whole logical sequence of proofs unreliable. Here is one (relatively obvious) example of incorrect reasoning. Warning: while the statement of the following theorem is correct, the proof itself is not correct, and I invite the reader to identify the wrong steps.

Theorem 2.1 (Steiner–Lehmus)**.** *A triangle with two equal angle bisects is isosceles.*

Proof. I start with the drawing. In $\triangle ABC$ it is given that the angle bisect AA_1 is equal to the angle

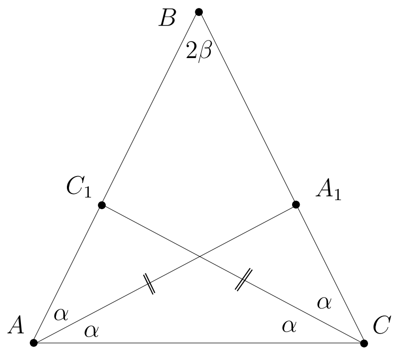

Figure 1: The drawing for Theorem 2.1.

bisect CC_1, therefore the triangles AA_1C and AC_1C are congruent by the side–angle–side comparison (two sides are equal by the given condition, one side is common, and the angles are the halves of the base angles). Therefore,

$$AC_1 = CA_1.$$

Next, in $\triangle ABA_1$ and $\triangle CC_1B$ $\angle AA_1B = \angle CC_1B$ since the other two angles are equal. Therefore these triangles are congruent by the angle–side–angle comparison (sides AA_1 and CC_1 are used respectively), and hence

$$C_1B = A_1B.$$

Summing two equalities together I get

$$AB = AC_1 + C_1B = CA_1 + A_1B = BC$$

as required. ∎

Math 478/678: History of Mathematics by Artem Novozhilov
e-mail: artem.novozhilov@ndsu.edu. Spring 2024

So, where is the incorrect step? I tried to hide it as good as I could, but it is hopefully clear that right in the second line of my proof I acidly assumed that the angles at the base are equal although it does not directly follow from the given conditions. Since we know that triangle is isosceles if and only if the angles at the base are equal, I pretty much *assumed* that my triangle is isosceles and *proved* that it is isosceles, hardly a very surprising fact given my logical development ($A \Rightarrow A$ is always true, no matter what the truth of the statement A).

Hence the conclusion: we must stop somewhere and assume some of the facts as true without proving them. Such fact are called *axioms* in mathematics. It was the greatest achievement of Euclid that he was able to come up a relatively short list of definitions, axioms, and common notions, which allowed him to put the whole body of mathematical knowledge on a sturdy foundation. I am not going to discuss the development of *Elements* in these notes, and instead refer the reader to the online version at `http://aleph0.clarku.edu/~djoyce/java/elements/bookI/bookI.html` where not only the English translation of Euclid's Elements is given, but also everything is given a thorough discussion from the point of view of modern standards of mathematical rigor.

One Euclid's axiom (namely, the fifth postulate) was especially important for the future development of the axiomatic method, which is discussed at length in the textbook. Euclid himself used his fifth postulate for the first time only in Proposition 29. In Proposition 32 it is proved (again, using the fifth postulate) that the sum of angles in any triangle sums to two right angles (i.e., to π radians). Do we have to use the fifth postulate? Not really, here is an example.

Theorem 2.2. *In any triangle the sum of the angles is equal to π.*

Proof. I start with an arbitrary triangle ABC and construct two triangles inside it by picking an arbitrary point on AC and connecting it with B (see Fig. 2). Let x be the angle measure of all the

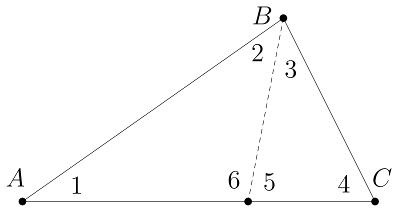

Figure 2: The drawing for Theorem 2.2.

angles in a triangle. I clearly have

$$\begin{aligned} \angle 1 + \angle 2 + \angle 6 &= x, \\ \angle 3 + \angle 4 + \angle 5 &= x, \\ \angle 1 + \angle 2 + \angle 3 + \angle 4 &= x, \end{aligned}$$

or

$$x + \angle 5 + \angle 6 = 2x.$$

Since angles 5 and 6 are adjacent, their sum is π, and hence

$$x = \pi$$

as stated. ■

So, is it possible to actually prove the sum of the angles of any triangle is π without the fifth postulate? The correct answer is "yes, if we replace it with an equivalent statement." Sometimes these equivalent statements are so much disquiet that it is difficult to see even for a trained eye. In this example, however, it is quite straightforward to recognize that the equivalent statement that was used is the fact that "all the triangles have the same angle measure," which can be shown to be equivalent to the fifth postulate (that it, the fifth postulate can be *proved* if one assumes this statement as true).

Finally, as it is discussed in the textbook (and even better through the link I have given above), the axiomatic system by Euclid and his proofs have a lot of shortcomings according to the modern mathematical standards. I will not discuss those issues here. Instead I opt to give you a proof, which is as rigorous as many proofs by Euclid, but clearly is incorrect. I will leave it to the student to figure out where I hide my mistake.

But before I start with the next "theorem," I would like to point out one very basic fact that will be used multiple times below. Namely, if I have a segment with two end points and if I build a perpendicular bisect to this segment then any point on this perpendicular is equidistant from the end points (see Fig. 3). The proof is immediate: find two triangles in this figure and note that by construction I have the side–angle–side comparison, and therefore these two triangles are congruent and therefore the sides that correspond to the distance from the point on the perpendicular to the end points are equal.

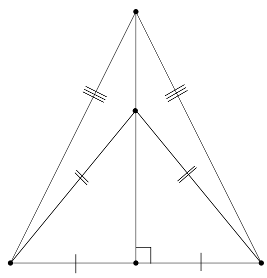

Figure 3: The distances from any point on the perpendicular bisect to the end points of the given segment are equal.

Now I am ready to state and prove (Warning: this is not a true theorem!)

Theorem 2.3.

$$1 = 0$$

Given: A quadrilateral $ABCD$ such that $\angle ABC = 91°$, $\angle BCD = 90°$, $AB = CD$.

Prove: $\angle ABC = \angle BCD$, and hence $1 = 0$.

Proof. To prove the theorem, I build two perpendicular bisects of the sides BC and AD and denote the point of their intersection as O (Fig. 4, left panel). I have that $BO = OC$ (property of perpendicular bisects), $AO = OD$ (the same reason), $AB = CD$ (by assumption). Hence $\triangle ABO = \triangle CDO$, and hence

$$\angle ABO = \angle OCD.$$

Since $\triangle BCO$ isosceles (perpendicular bisects!),

$$\angle OBC = \angle BCO.$$

Since $\angle ABC = \angle ABO + \angle OBC$ and $\angle BCD = \angle OCD + \angle BCO$, the required conclusion follows:

$$\angle ABC = \angle BCD.$$

Ok, stop for a second, this cannot be true. Probably my drawing is not correct. What else is possible? Maybe the perpendicular bisects intersect exactly on the side AD (Fig. 4, right panel). Now I have, however, $AO = OD$ by construction, $AB = CD$ by the initial conditions, $BO = CO$ by the properties of the perpendicular bisect, and hence $\angle ABO = \angle OCD$, $\angle OBC = \angle BCO$ as the base angles in an isosceles triangle, and therefore the same conclusion follows:

$$\angle ABC = \angle BCD.$$

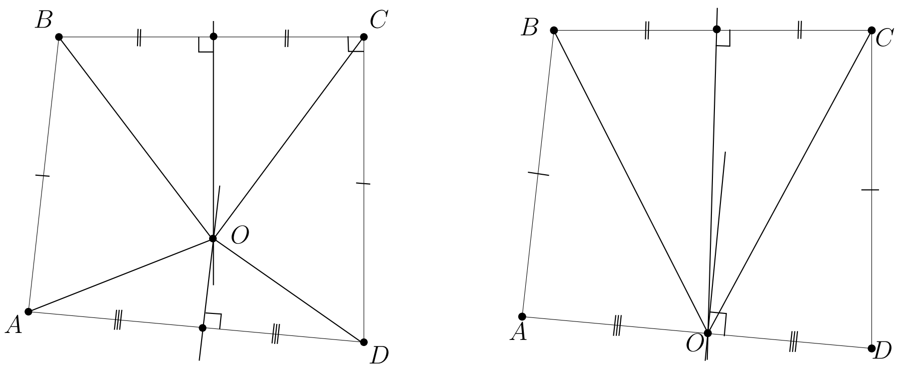

Figure 4: Drawings for the first two cases in Theorem 2.3.

Wait, these are certainly not all the possible cases. Indeed, I can assume that my perpendicular bisects intersect as in Fig. 5, left panel. Ant yet here, for exactly the same reasons, I can conclude that $\triangle ABO = \triangle OCD$, and hence $\angle ABO = \angle OCD$. Moreover, again, $\angle OBC = \angle OCB$ as the base angles in isosceles triangle, and since $\angle ABC = \angle ABO - \angle OBC$ and $\angle BCD = \angle OCD - \angle OCB$, the same conclusion follows:

$$\angle ABC = \angle BCD.$$

What else is possible? Maybe these two perpendicular bisects intersect below AD? I leave it as an exercise to check that very similar arguments lead to the same conclusion.

Ok, maybe these bisects do not intersect at all (Fig. 5, right panel)? In this case I must conclude that BC and AD are parallel, and hence $ABCD$ is either trapeze or parallelogram. If it is a trapeze the fact that $AB = CD$ immediately implies that $\angle ABC = \angle BCD$ (think this out), and if it is a parallelogram, the sum of the angles at the same side must be $180°$, or, in other words, I end up with the same equality

$$1 = 0.$$

■

2.2 Spherical geometry as the simplest non-euclidian geometry

2.3 Further reading

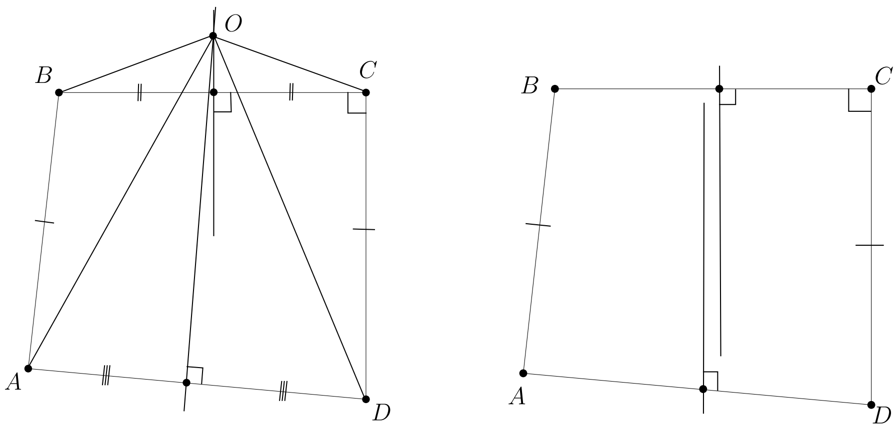

Figure 5: Drawings for the next two cases in Theorem 2.3.